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Research Article

Structural Stability of Planar Bimodal Linear Systems

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Structural stability ensures that the qualitative behavior of a system is preserved under small perturbations. We study it for planar bimodal linear dynamical systems, that is, systems consisting of two linear dynamics acting on each side of a given hyperplane and assuming continuity along the separating hyperplane. We describe which one of these systems is structurally stable when (real) spiral does not appear and when it does we give necessary and sufficient conditions concerning finite periodic orbits and saddle connections. In particular, we study the finite periodic orbits and the homoclinic orbits in the saddle/spiral case.

1. Introduction

Structural stability ensures that the qualitative behavior of a system is preserved under small perturbations: a system is structurally stable if anyone in some neighborhood is equivalent to it (in particular, they have the same dynamical behavior). We study this property for a class of piecewise linear systems. Piecewise linear systems have attracted the interest of the researchers in recent years by their wide range of applications, as well as the possible theoretical approaches. See, for example, [1–8]. In particular, bimodal linear systems consist of two subsystems acting on each side of a given hyperplane, assuming continuity along the separating hyperplane. We focus on the planar case. Indeed, it is very commonly found in applications (see the above references).

As we have pointed out, a definition of structural stability involves a topology in the set of the considered systems (which defines the "small perturbations") and an equivalence relation (which defines the "preservation of the behavior"). For piecewise linear systems, the natural topology is the one of the Euclidean space formed by the coefficients of the matrices which determine each subsystem. Concerning the equivalence relation, there are some different natural options. For example, for single linear systems, those having positive trace and positive determinant form a unique C^0-class, whereas they are partitioned in four C^1-classes (spirals, nodes, improper nodes, and starred nodes). Anyway, when a topology and an equivalence relation are fixed, the structural stability points are those belonging to an open equivalence class.

Alternative approaches are possible. For example, in [9], one asks about generic properties, which are verified by "almost all" piecewise linear systems. From a topological point of view, it is a matter of density instead of openness. Indeed, the properties there are both generic and stable. Also, Arnold's techniques [10] can be partially applied because although the equivalence relation is not defined by the action of a Lie group, the equivalence classes are probably differentiable manifolds.

Here, we focus on structural stability in the sense in [11], where a list of necessary and sufficient conditions is given for planar piecewise linear systems. Our aim is to specify these criteria in terms of the coefficients of the matrices, in the particular case of bimodal linear systems. The first step is collected in Theorem 6. However, further specific studies are necessary in several cases. As a second step, we tackle (Theorem 7) the existence of homoclinic orbits and finite periodic orbits in the saddle/spiral case. It allows us (Corollary 11) to ensure its structural stability for certain values of the parameters. We expect that, for bimodal systems, a full characterization of the structural stability in terms of the coefficients of the matrices is possible.

Even more, we expect that also a systematic study of the bifurcations is possible. Bifurcations are the frontier points of an open class, so that they come out of their class by small perturbations. Again, it depends on the considered equivalence relation. For example, the improper nodes and the starred nodes are C^1-bifurcation between spirals and nodes but not C^0-bifurcation because all of them are C^0-equivalent. Indeed, the C^1-frontier of spirals/nodes is stratified as follows: a 1-codimensional manifold formed by the improper nodes and a 3-codimensional manifold formed by the starred nodes. (Hence, improper nodes appear generically in 1-parameterized families of linear systems, whereas starred nodes appear only in 3-parameterized families.) Here, three bifurcations are presented in Corollary 11: 1-codimensional (two of them) and 2-codimensional (the third one).

In Section 3, we adapt the conditions stated in [11] for piecewise linear planar dynamical systems to the particular class of bimodal ones. We conclude that if some subsystem is a starred node, a center, or a degenerate node, then the bimodal system is not structurally stable. Moreover, we list the remaining possible cases, and we ensure that the bimodal system is structurally stable if none of the subsystems is a (real) spiral. The other cases need further specific analysis.

In particular, when a (real) spiral appears, it is necessary to study the finite periodic orbits and the homoclinic orbits. In Section 4, we study the structural stability of bimodal systems for the saddle/spiral case. We conclude that this bimodal system is structurally stable if $0 < \gamma_1 < \gamma_H$, where γ_1 is the trace of the spiral matrix and γ_H is the only value where a homoclinic orbit appears. The study will be continued in future works (see [12].)

Throughout the paper, $\mathbf{R}$ will denote the set of real numbers, $M_{n\times m}(\mathbf{R})$ the set of matrices having n rows and m columns and entries in $\mathbf{R}$ (in the case where $n = m$, we will simply write $M_n(\mathbf{R})$), and $Gl_n(\mathbf{R})$ the group of nonsingular matrices in $M_n(\mathbf{R})$. Finally, we will denote by $e_1, \ldots, e_n$ the natural basis of the Euclidean space $\mathbf{R}^n$.

2. Structurally Stable BLDS: Definitions

We consider

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + B_1 \quad \text{if } Cx(t) \le 0, \\ \dot{x}(t) &= A_2 \mathbf{x}(t) + B_2 \quad \text{if } Cx(t) \ge 0, \end{aligned} \tag{1}$$

where $A_1, A_2 \in M_n(\mathbf{R})$; $B_1, B_2 \in M_{n\times 1}(\mathbf{R})$; $C \in M_{1\times n}(\mathbf{R})$. We assume that the dynamic is continuous along the separating hyperplane $H = \{x \in \mathbf{R}^n : Cx = 0\}$; that is to say, both subsystems coincide with $Cx(t) = 0$.

By means of a linear change in the state variable $x(t)$, we can consider $C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \in M_{1\times n}(\mathbf{R})$. Hence, $H = \{x \in \mathbf{R}^n : x_1 = 0\}$ and continuity along H is equivalent to

$$B_2 = B_1, \qquad A_2 e_i = A_1 e_i, \quad 2 \le i \le n. \tag{2}$$

We will write from now on $B = B_1 = B_2$.

Definition 1. In the above conditions, one says that the triplet of matrices (A_1, A_2, B) defines a bimodal linear dynamical system (BLDS).

Table 1: Critical points classification.

Spiral	$a_{21} = 0$, λ_1, λ_2 conjugate complex numbers
Saddle	$a_{21} = 0$, $\lambda_1 \cdot \lambda_2 < 0$
Node	$a_{21} = 0$, $\lambda_1 \cdot \lambda_2 > 0, \lambda_1 \ne \lambda_2$
Starred node	$a_{21} = 0$, $\lambda_1 = \lambda_2 = \lambda$, $\lambda \ne 0, \lambda \in \mathbf{R}$
Improper node	$a_{21} = 1$, $\lambda_1 = \lambda_2 = \lambda$, $\lambda \ne 0, \lambda \in \mathbf{R}$
Degenerate node	$a_{21} = 0$, $\lambda_1 = \lambda, \lambda_2 = 0$, $\lambda \ne 0, \lambda \in \mathbf{R}$

The placement of the equilibrium points will play a significative role in the dynamics of a BLDS. So, one defines the following.

Definition 2. Let one assume that a subsystem of a BLDS has a unique equilibrium point, not lying in the separating hyperplane. One says that this equilibrium point is real if it is located in the half-space corresponding to the considered subsystem. Otherwise, one says that the equilibrium point is virtual.

It is clear that not any pair of equilibrium points are compatible. For example, two real saddles are not possible. (Table 1 lists the compatible pairs, excluding centers, starred nodes, and degenerate nodes.)

Our goal is to characterize the planar BLDS which are structurally stable in the sense of [11].

Definition 3. A triplet of matrices (A_1, A_2, B) defining a BLDS is said to be (regularly) structurally stable if it has a neighborhood $V(A_1, A_2, B)$ such that, for every $(A_1', A_2', B') \in V(A_1, A_2, B)$, there is a homeomorphism of $\mathbf{R}^2$ preserving the hyperplane H which maps the oriented orbits of (A_1', A_2', B') into those of (A_1, A_2, B) and it is differentiable when restricted to finite periodic orbits.

A natural tool in the study of BLDS is simplifying the matrices A_1, A_2, B by means of changes in the variables $x(t)$ which preserve the qualitative behavior of the system (in particular, the condition of structural stability). See [5] for some partial results and [13] for a systematic obtention of reduced forms. So, we consider linear changes in the state variables space preserving the hyperplanes $x_1(t) = k$, which will be called *admissible basis changes*. Thus, they are basis changes given by a matrix $S \in Gl_n(\mathbf{R})$,

$$S = \begin{pmatrix} 1 & 0 \\ U & T \end{pmatrix}, \quad T \in Gl_{n-1}(\mathbf{R}), \ U \in M_{n-1\times 1}(\mathbf{R}). \tag{3}$$

Also, translations parallel to the hyperplane H are allowed.

3. Structurally Stable Planar BLDS: General Criteria

Let us consider a planar BLDS. For each subsystem, we follow the terminology in [14] according to its Jordan reduced form, except for the "focus" which we have substituted by the denomination "starred node." Here, we reproduce this classification.

If we denote by $\begin{pmatrix} \lambda_1 & 0 \\ a_{21} & \lambda_2 \end{pmatrix}$ the reduced matrix, then we identify critical points classification as shown in Table 1.

For the particular case of BLDS, the general conditions in [11] in order to be structurally stable can be simplified as follows.

Corollary 4. *A planar BLDS is structurally stable, if and only if the following conditions hold.*

(1) *Singularities conditions:*

(a) *all its singularities at infinity are disjoint from the separating axis;*
(b) *all its singularities at infinity are hyperbolic;*
(c) *all its finite singularities are disjoint from the separating axis;*
(d) *all its finite singularities are hyperbolic;*
(e) *all its tangencies with the separating axis are isolated.*

(2) *Periodic orbits conditions:*

(a) *all its finite periodic orbits are not tangent to the separating axis;*
(b) *all its finite periodic orbits are hyperbolic;*
(c) *the infinite periodic orbit at infinity is hyperbolic.*

(3) *There are no finite orbits which joint either*

(a) *two different saddle points ("saddle-saddle orbits"),*
(b) *a saddle point with itself ("saddle-loop orbits" or "homoclinic orbits"),*
(c) *a saddle point and a tangency ("saddle-tangency orbits").*

We will specify these conditions for a triplet of matrices defining a planar BLDS. We begin with conditions (1)(a) and (1)(c) in Corollary 4.

Lemma 5. *The triplets of matrices representing a structurally stable planar BLDS can be reduced to the form*

$$A_1 = \begin{pmatrix} a_1 & 1 \\ a_2 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \gamma_1 & 1 \\ \gamma_2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}, \quad b_2 \neq 0. \tag{4}$$

Proof. Given a planar BLDS defined by a triplet $(\overline{A}_1, \overline{A}_2, \overline{B})$, such as

$$\overline{A}_1 = \begin{pmatrix} \overline{a}_1 & \overline{a}_3 \\ \overline{a}_2 & \overline{a}_4 \end{pmatrix}, \qquad \overline{A}_2 = \begin{pmatrix} \overline{\gamma}_1 & \overline{a}_3 \\ \overline{\gamma}_2 & \overline{a}_4 \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} \overline{b}_1 \\ \overline{b}_2 \end{pmatrix}, \tag{5}$$

the condition (1)(a) in Corollary 4 is equivalent to

$$\begin{pmatrix} \overline{a}_1 & \overline{a}_3 \\ \overline{a}_2 & \overline{a}_4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6}$$

which gives $\overline{a}_3 \neq 0$. As it is proved in [13], when $\overline{a}_3 \neq 0$, by means of a suitable admissible basis change, the triplet can be reduced to

$$\begin{pmatrix} a_1 & 1 \\ a_2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \gamma_1 & 1 \\ \gamma_2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{7}$$

Moreover, by means of the translation $\overline{x}_2 = x_2 + b_1$, we obtain

$$\begin{pmatrix} a_1 & 1 \\ a_2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \gamma_1 & 1 \\ \gamma_2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ b_2 \end{pmatrix}. \tag{8}$$

Then, the condition (1)(c) in Corollary 4 is equivalent to

$$\begin{pmatrix} a_1 & 1 \\ a_2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_{2e} \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{9}$$

which gives $b_2 \neq 0$. □

Now, we apply the remaining conditions in Corollary 4.

Theorem 6. *Let one consider a planar BLDS as in Lemma 5.*

(1) *The only tangency (i.e., $\dot{x}_1(0, x_2) = 0$) is the origin $(0, 0)$.*

(2) *If one of the subsystems is a center, a degenerate node, or a starred node, then the BLDS is not structurally stable.*

More in general, the only BLDS verifying (1)(a), (1)(c), and (1)(d) in Corollary 4 are those in Table 2.

(3) *The cases 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, and 16 (those where none of the subsystems is a real spiral) are structurally stable.*

(4) *In case 3, it is structurally stable, if and only if*

(a) *the finite periodic orbits are hyperbolic,*
(b) *there are no saddle-loop orbits,*
(c) *there are no finite orbits connecting a saddle and a tangency point.*

(5) *In the cases 7, 11, and 15, the BLDS is structurally stable, if and only if the above condition (4)(a) holds.*

Proof. We proceed with the same order for the proof.

(1) Clearly, $\dot{x}_1(0, x_2) = 0$ implies $x_2 = 0$.

(2) Starred nodes have been excluded by (1)(a). The condition (1)(d) excludes degenerate nodes and centers; that is to say, for $i = 1, 2$,

$$\det A_i \neq 0, \qquad \text{if } \det A_i > 0, \text{ then } \operatorname{trace} A_i \neq 0. \tag{10}$$

Therefore, the subsystems have a unique equilibrium point

$$\left(-\frac{b_2}{a_2}, a_1 \frac{b_2}{a_2} \right), \qquad \left(-\frac{b_2}{\gamma_2}, \gamma_1 \frac{b_2}{\gamma_2} \right). \tag{11}$$

TABLE 2: Enumeration of the possible cases.

Subsystem 1/Subsystem 2	Virtual saddle	Real node	Real spiral	Real improper node
Real saddle	1 ($b_2 > 0$)	2 ($b_2 > 0$)	3 ($b_2 > 0$)	4 ($b_2 > 0$)
Virtual node	5 ($b_2 < 0$)	6 ($b_2 > 0$)	7 ($b_2 > 0$)	8 ($b_2 > 0$)
Virtual spiral	9 ($b_2 < 0$)	10 ($b_2 < 0$)	11 ($b_2 > 0$)	12 ($b_2 > 0$)
Virtual improper node	13 ($b_2 < 0$)	14 ($b_2 < 0$)	15 ($b_2 < 0$)	16 ($b_2 > 0$)

But not all combinations (real/virtual) are possible. For example, let us assume a real saddle as subsystem 1 (i.e., $a_2 > 0$ and $b_2 > 0$). Then, the subsystem 2 must be, for $\gamma_2 > 0$, a virtual saddle and, for $\gamma_2 < 0$, a real node, a real improper node, or a real spiral. They are the cases 1, 2, 4, and 3, respectively. Analogously for the remainder cases of the table.

(3) As starred nodes and degenerate nodes are excluded, all the BLDS in Table 2 satisfy condition (1)(b) in Corollary 4. Moreover, the feature that the only tangency point is the origin implies the conditions (1)(e) and (2)(a) (we notice that centers are excluded). Concerning (2)(c), it is clear that only in the case 11 there exists infinite periodic orbit at infinity and that it is hyperbolic (its character of attracting/repelling depends on the signs of the real and imaginary part of the complex eigenvalues). Obviously, condition (3)(a) makes no sense in Table 2.

Again, the remaining conditions (2)(b), (3)(b), and (3)(c) make no sense in the cases listed in point 3 of Theorem 6, so that they are structurally stable.

(4) On the other hand, they must be verified in case 3: we adapt them to (a), (b), and (c) in point 4 of Theorem 6.

(5) Finally, in the cases in point 5, only (a) must be verified, because again (b) and (c) make no sense. □

4. Structurally Stable Planar BLDS: Specific Studies

Theorem 6 collects the conclusions of applying to planar BLDS the general criteria in [11] for a piecewise linear system to be structurally stable. Nevertheless, cases 3, 7, 11, and 15 need additional specific studies. For example, see in [15, 16] partial results concerning case 7. In this section, we focus on conditions (a) and (b) of case 3 for divergent spirals, leaving the remaining cases for future works.

Thus, let us assume a BLDS as in Lemma 5, verifying the following.

(i) The left subsystem is a real saddle; that is, $a_2 > 0$, $b_2 > 0$. In particular, its equilibrium point is $(-b_2/a_2, a_1(b_2/a_2))$, and the invariant manifold cuts the separating line at $(0, -b_2/\lambda_2)$ and $(0, -b_2/\lambda_1)$, where $\lambda_2 < 0 < \lambda_1$ are the eigenvalues of A_1. (Consider $\lambda_1 + \lambda_2 = a_1, \lambda_1\lambda_2 = -a_2$.)

(ii) The right subsystem is a real divergent spiral; that is, $\gamma_1 > 0$, $\gamma_2 < 0$, and $\gamma_1^2 < -4\gamma_2$. In particular, its equilibrium point is $(-b_2/\gamma_2, \gamma_1(b_2/\gamma_2))$. We write $\alpha \pm i\beta$, $\beta > 0$, the eigenvalues of A_2. (Consider $2\alpha = \gamma_1, \alpha^2 + \beta^2 = -\gamma_2$.)

Theorem 7. *As above, let one assume*

$$b_2 > 0, \qquad a_2 > 0, \qquad \gamma_1 > 0,$$
$$\gamma_2 < 0, \qquad \gamma_1^2 < -4\gamma_2 \tag{12}$$

and let

$$\lambda_2 < 0 < \lambda_1 \text{ the eigenvalues of } A_1,$$
$$\alpha \pm i\beta, \quad \beta > 0, \text{ the eigenvalues of } A_2. \tag{13}$$

In addition, let $M > 0$ and $0 < \varphi < \pi$ defined by

$$M\cos(\varphi) = \alpha - \frac{\alpha^2 + \beta^2}{\lambda_2}, \qquad M\sin(\varphi) = \beta. \tag{14}$$

Then, consider the following.

(1)

(a) *If $a_1 > 0$, then there is no homoclinic orbit.*

(b) *If $a_1 = 0$, then there is a homoclinic orbit only for $\gamma_1 = 0$, which is not a considered case.*

(c) *If $a_1 < 0$, the only homoclinic (i.e., saddle-loop) orbit appears for the value γ_H of γ_1 verifying*

$$t = \frac{1}{\gamma_1}\ln\left(\frac{\lambda_2^2}{\lambda_1^2}\frac{\lambda_1^2 - \gamma_1\lambda_1 - \gamma_2}{\lambda_2^2 - \gamma_1\lambda_2 - \gamma_2}\right) \tag{15}$$

being

$$\exp(\alpha t)\sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \le \beta t \le \frac{3\pi}{2} + \varphi. \tag{16}$$

Moreover, $\gamma_H > a_1\gamma_2/a_2$.

(2)

(a) *If $a_1 > 0$, then there are no finite periodic orbits.*

(b) *If $a_1 = 0$, then there are finite periodic orbits (all of them) only for $\gamma_1 = 0$, which is not a considered case.*

(c) *If $a_1 < 0$, at least a finite periodic orbit appears for $0 < \gamma_1 < \gamma_H$, all of the finite periodic orbits being hyperbolic and disjoint from the tangency points. No saddle-tangency orbits appear.*

The proof is based on the following lemmas.

Lemma 8. *A spiral cuts $x_1 = 0$ in x_{21} and x_{22}, if and only if*

$$\exp(\mu t) = \frac{b_2 + \mu x_{22}}{b_2 + \mu x_{21}}, \tag{17}$$

where $\mu = \alpha + i\beta$.

Proof. The solution of the system for the spiral is

$$\begin{aligned}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= \begin{pmatrix} \mu & \overline{\mu} \\ \gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \exp(\mu t) & 0 \\ 0 & \exp(\overline{\mu} t) \end{pmatrix} \begin{pmatrix} \mu & \overline{\mu} \\ \gamma_2 & \gamma_2 \end{pmatrix}^{-1} \\ &\times \begin{pmatrix} x_1(0) + \dfrac{b_2}{\gamma_2} \\ x_2(0) - \gamma_1 \dfrac{b_2}{\gamma_2} \end{pmatrix} + \begin{pmatrix} -\dfrac{b_2}{\gamma_2} \\ \gamma_1 \dfrac{b_2}{\gamma_2} \end{pmatrix},\end{aligned} \tag{18}$$

where $\overline{\mu}$ is the conjugate of the eigenvalue μ.

Considering that the starting and final point have $x_1 = 0$ and denoting $x_2(0) = x_{21}$ and $x_2(t) = x_{22}$, we get

$$\begin{aligned}&\gamma_2(\mu - \overline{\mu}) \begin{pmatrix} \dfrac{b_2}{\gamma_2} \\ x_{22} - \gamma_1 \dfrac{b_2}{\gamma_2} \end{pmatrix} \\ &= \begin{pmatrix} \mu \exp(\mu t) & \overline{\mu} \exp(\overline{\mu} t) \\ \gamma_2 \exp(\mu t) & \gamma_2 \exp(\overline{\mu} t) \end{pmatrix} \begin{pmatrix} b_2 - \overline{\mu}\left(x_{21} - \gamma_1 \dfrac{b_2}{\gamma_2}\right) \\ -b_2 + \mu\left(x_{21} - \gamma_1 \dfrac{b_2}{\gamma_2}\right) \end{pmatrix}.\end{aligned} \tag{19}$$

Multiplying both sides of the system by $(\gamma_2 \ \ -\overline{\mu})$, we obtain

$$\exp(\mu t) = \frac{b_2 - \overline{\mu} x_{22} + \overline{\mu}\gamma_1 (b_2/\gamma_2)}{b_2 - \overline{\mu} x_{21} + \overline{\mu}\gamma_1 (b_2/\gamma_2)} \tag{20}$$

which is equivalent to

$$\exp(\mu t) = \frac{b_2 + \mu x_{22}}{b_2 + \mu x_{21}}. \tag{21}$$

□

Lemma 9. *Let one consider the saddle-spiral orbit passing through $(0, -b_2/\lambda_2)$. Then, its first intersection with the separating hyperplane (if it exists) is determined by*

$$\exp(\alpha t)\sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \leq \beta t \leq \frac{3\pi}{2} + \varphi. \tag{22}$$

Proof. Using Lemma 8, imposing that $x_{21} = -b_2/\lambda_2$, we get

$$\begin{aligned}x_{22} = \frac{b_2}{\alpha^2 + \beta^2} &\left[(M \exp(\alpha t)\cos(\beta t - \varphi) - \alpha)\right. \\ &\left. + i\,(M \exp(\alpha t)\sin(\beta t - \varphi) + \beta)\right].\end{aligned} \tag{23}$$

□

Lemma 10. *Let one assume that a finite periodic orbit exists. Then,*

$$A^+\gamma_1 = -A^- a_1, \tag{24}$$

where A^+ and A^- are the enclosed areas in the right and the left side, respectively.

Proof. An analogous result is proved in [16] by means of Green's formula. Alternatively, here, we follow the approach in [17]. Let us consider the following continuous energy function

$$\begin{aligned}E^- &= \frac{1}{2}(a_2 x_1 + b_2)^2 - \frac{1}{2}x_2^2 \quad \text{if } x_1 \leq 0, \\ E^+ &= \frac{1}{2}(\gamma_2 x_1 + b_2)^2 - \frac{1}{2}x_2^2 \quad \text{if } x_1 \geq 0.\end{aligned} \tag{25}$$

If we consider the following energy piecewise function, the change in energy ΔE along a periodic orbit must be null. Hence,

$$\begin{aligned}0 = \Delta E &= E^+\left(0, x_2^m\right) - E^+\left(0, x_2^M\right) \\ &\quad + E^-\left(0, x_2^M\right) - E^-\left(0, x_2^m\right) \\ &= \int_{x_2^M}^{x_2^m} \frac{dE^+}{dx_2} dx_2 + \int_{x_2^m}^{x_2^M} \frac{dE^-}{dx_2} dx_2,\end{aligned} \tag{26}$$

where x_2^M and x_2^m are the top and the bottom intersections with the separating hyperplane, respectively.

But (25) implies

$$\frac{dE^+}{dx_2} = (\gamma_2 x_1 + b_2)\frac{dx_1}{dx_2} - x_2 \tag{27}$$

and from the equations of the bimodal system we have

$$\frac{dx_1}{dx_2} = \frac{\gamma_1 x_1 + x_2}{\gamma_2 x_1 + b_2}, \tag{28}$$

so

$$\frac{dE^+}{dx_2} = \gamma_1 x_1. \tag{29}$$

And the first integral can be computed by

$$\int_{x_2^M}^{x_2^m} \frac{dE^+}{dx_2} dx_2 = -\gamma_1 A^+, \tag{30}$$

where A^+ is the enclosed areas in the right side. And analogously for the other integral.

Thus, $\Delta E = 0$, if and only if

$$A^+\gamma_1 = -A^- a_1. \tag{31}$$

□

Proof of Theorem 7. (1)(b), (2)(b) For $a_1 = \gamma_1 = 0$, it is obvious that we have a saddle/center (a not considered case), being both subsystems symmetric with regard to the axis

$x_2 = 0$. Then, we have a homoclinic orbit (the first intersection with $x_1 = 0$ of the orbit passing through $(0, -b_2/\lambda_2)$ is just $(0, -b_2/\lambda_1)$) and all the orbits inside it are finite periodic orbits.

It is also clear that if γ_1 increases (being $a_1 = 0$), then the orbit passing through $(0, -b_2/\lambda_2)$ cuts the axis $x_1 = 0$ below $(0, -b_2/\lambda_1)$ (notice that the spirals become divergent and the equilibrium point descends). So the homoclinic and the finite periodic orbits disappear.

(1)(a), (2)(a) If, in addition, a_1 increases, then the point $(0, -b_2/\lambda_1)$ ascends.

(1)(c) Using Lemma 9, imposing that its first intersection with the separating hyperplane cuts at $(0, -b_2/\lambda_1)$,

$$\exp(\alpha t)\sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \le \beta t \le \frac{3\pi}{2} + \varphi \tag{32}$$

being

$$\frac{b_2}{\alpha^2 + \beta^2}\left(M \exp(\alpha t)\cos(\beta t - \varphi) - \alpha\right) = -\frac{b_2}{\lambda_1}, \tag{33}$$

we get

$$t = \frac{1}{\gamma_1}\ln\left(\frac{\lambda_2^2}{\lambda_1^2}\frac{\lambda_1^2 - \gamma_1\lambda_1 - \gamma_2}{\lambda_2^2 - \gamma_1\lambda_2 - \gamma_2}\right). \tag{34}$$

Moreover, for the existence of the homoclinic orbit, it must be verified that

$$\frac{\lambda_2^2}{\lambda_1^2}\frac{\lambda_1^2 - \gamma_H\lambda_1 - \gamma_2}{\lambda_2^2 - \gamma_H\lambda_2 - \gamma_2} > 1 \tag{35}$$

which is equivalent to

$$\gamma_H a_2 \lambda_2 - \gamma_2\lambda_2^2 > \gamma_H a_2 \lambda_1 - \gamma_2 \lambda_1^2 \tag{36}$$

and, from it, we get

$$\gamma_H > \frac{a_1\gamma_2}{a_2}. \tag{37}$$

(2)(c) For $0 < \gamma_1 < \gamma_H$, we claim that there is at least a stable limit cycle.

The first step is to show that a periodic solution exists. We use the classical argument of Poincaré (as, e.g., in [17]). Consider a trajectory that starts at height x_2 on the right side of the separating hyperplane, crosses to the left side, and intersects the left side of the same hyperplane at some new height $P(x_2)$. The mapping from x_2 to $P(x_2)$ is called the Poincaré map. It tells us how the height of a trajectory changes after one lap. If we can show that there is a point x_2^* such that $P(x_2^*) = x_2^*$, then the corresponding trajectory will be a periodic orbit and it is stable if $|P'(x_2)| < 1$. Being linear, both subsystems, $P(x_2)$, can be easily computed. See, for example, Figure 1. In order to show that such a x_2^* must exist, it is sufficient to know what the graph of $P(x_2)$ looks like, roughly.

Let us consider $0 \le x_2 \le -b_2/\lambda_2$. For $x_2 = 0$, we have the unique tangency trajectory so that the first

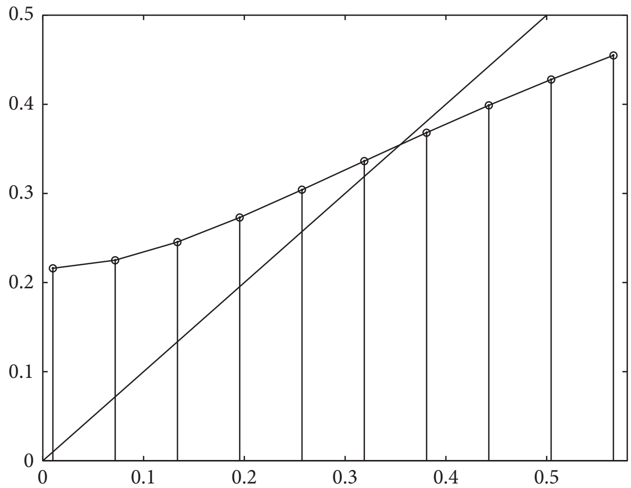

Figure 1: Distances between original and Poincaré image points. Intersection with the bisectrix corresponds to the stable limit cycle. Parameter values are the ones used for Example (1).

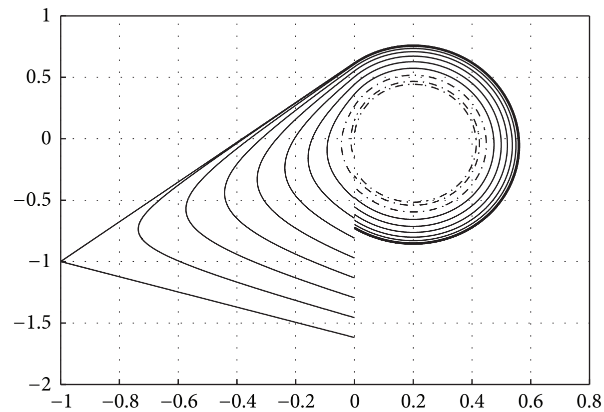

Figure 2: Appearance of a finite periodic orbit in case 3: $a_1 = -1$, $a_2 = 1$, $\gamma_1 = 0.1$, $\gamma_2 = -5$, and $b_2 = 1$.

intersection with the separating line is under $(0,0)$ and finally $P(0) > 0$. On the other hand, for $x_2 = -b_2/\lambda_2$, the first intersection is upper $(0, -b_2/\lambda_1)$ (recall $\gamma_1 < \gamma_H$), so that $P(-b_2/\lambda_2) < -b_2/\lambda_2$. Furthermore, $P(x_2)$ is a continuous function (from the theorem about the dependence of the solutions on initial conditions) and indeed it is a smooth and monotonic function (if not, two trajectories would cross). So, by the intermediate value theorem, the graph of $P(x_2)$ must cross the bisectrix somewhere; that intersection is our desired x_2^*.

We must exclude the possibility that $P(x_2) \equiv x_2$ on some interval, in which case there would be a band of infinitely many closed orbits. If it happens, Lemma 10 ensures that A^+/A^- is constant $(= -a_1/\gamma_1)$ in this interval; but, being A^+, A^- analytic functions, the quotient will be constant everywhere, which is obviously false (e.g., when $x_2 \to 0$).

Finally, as $P(x_2^*) = x_2^*$ is an isolated crossing and $P(x_2^*)$ is monotonic increasing, then $0 < P'(x_2) < 1$ so that the periodic orbit is an attractor orbit.

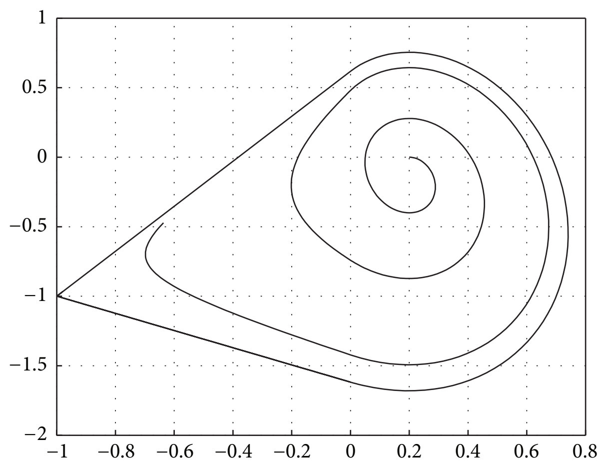

FIGURE 3: Appearance of a homoclinic orbit in case 3: $a_1 = -1$, $a_2 = 1$, $\gamma_1 = \gamma_H = 0.742$, $\gamma_2 = -5$, and $b_2 = 1$.

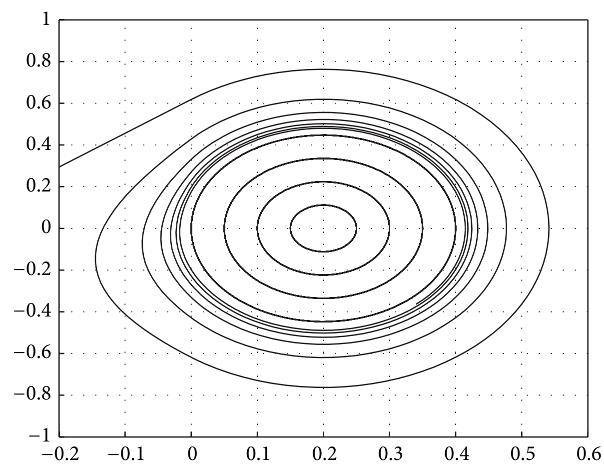

FIGURE 4: Appearance of nonhyperbolic periodic orbits in case 3: $a_1 = -1$, $a_2 = 1$, $\gamma_1 = 0$, $\gamma_2 = -5$, and $b_2 = 1$.

Other values x_2 verifying $P(x_2) = x_2$ can appear. But the above reasoning shows that the possibility that $P(x_2) \equiv x_2$ on some interval is excluded, so that the corresponding periodic orbit is again hyperbolic. □

Corollary 11. *The systems in Theorem 7 with $a_1 < 0$ and $0 < \gamma_1 < \gamma_H$ are structurally stable.*

Bifurcations appear for

(i) $a_1 < 0$, $\gamma_1 = \gamma_H$: *homoclinic orbit,*

(ii) $a_1 < 0$, $\gamma_1 = 0$: *nonhyperbolic finite periodic orbits,*

(iii) $a_1 = 0$, $\gamma_1 = 0$: *both kinds of orbits.*

Proof. The existence of a periodic orbit implies that (3)(c) in Corollary 4 is verified. Hence, this case verifies all the conditions in that corollary. □

Examples. (1) We show the structurally stable case: $a_1 = -1$, $a_2 = 1$, $\gamma_1 = 0.1$, $\gamma_2 = -5$, and $b_2 = 1$ in Figure 2. We plot the phase portrait corresponding to the Poincaré map on the section $x = 0$ for different initial points: for each of them, the orbits are integrated until the next oriented cut. The continuous lines correspond to inward spiraling orbits and the discontinuous lines to outward spiraling ones. A hyperbolic finite periodic orbit exists between them.

(2) Bifurcations are as follows:

(i) homoclinic orbit (Figure 3): $a_1 = -1$, $a_2 = 1$, $\gamma_1 = \gamma_H$, $\gamma_2 = -5$, and $b_2 = 1$,

(ii) nonhyperbolic periodic orbits (Figure 4): $a_1 = -1$, $a_2 = 1$, $\gamma_1 = 0$, $\gamma_2 = -5$, and $b_2 = 1$,

(iii) both kinds of orbits (Figure 5): $a_1 = 0$, $a_2 = 1$, $\gamma_1 = 0$, $\gamma_2 = -5$, and $b_2 = 1$.

Remark 12. In [12], some partial results for $\gamma_1 < 0$ and $\gamma_1 > \gamma_H$ have been presented.

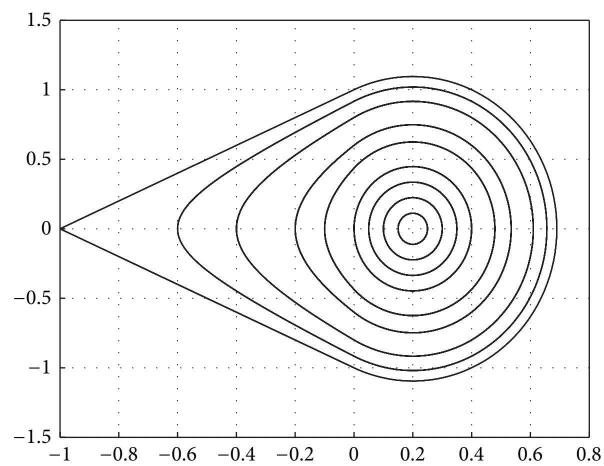

FIGURE 5: Appearance of both kinds of singularities in case 3: $a_1 = 0$, $a_2 = 1$, $\gamma_1 = 0$, $\gamma_2 = -5$, and $b_2 = 1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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